PROPAGATION OF MAGNETOACOUSTIC WAVES IN STRATIFIED MEDIA

L. Ya. Kosachevskii

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We shall consider the reflection of magnetoacoustic waves from a plane-parallel layer of an electrically conducting liquid or gas in a constant uniform magnetic field **H**. Expressions will be established for the reflection and transmission coefficients of the layer in the limiting cases of weak and strong magnetic fields.

§1. Suppose that a fast magnetoaccoustic wave (Fig. 1) is incident on a plane-parallel layer of thickness d at an arbitrary angle. The boundary of the layer lies in the xy plane, while the plane of incidence lies in the xz plane. The latter plane also includes the vector **H** which is at an angle φ to the x axis. The media on either side of the layer are electrically conducting. Both fast and slow magnetoacoustic waves will then be reflected into the upper medium and transmitted to the lower medium. As a result of multiple reflections at the boundaries of the layer there will be a resultant field of magnetoacoustic waves of both types propagating in opposite directions along the z-axis.

The velocity vectors and the variations of the magnetic field in the magnetoacoustic waves will lie in the plane of incidence. The Alfven waves are polarized at right-angles to this plane and propagate independently of the magnetoacoustic waves. We therefore need not take them into account in the present case.

The upper medium, the layer, and the lower medium will be indicated by numbers 3, 2, and 1, respectively. Quantities referring to various types of magnetoacoustic wave will be indicated by letters with two subscripts, for example, $A_{\mu\nu}$, where μ = 1, 2, 3 is the number of the medium in which the wave is propagating, and $\nu = 1$ and $\nu = 2$ refer to fast and slow waves, respectively. Moreover, quantities referring to waves propagating in the positive direction of the z-axis will be indicated by primes.

The magnetohydrodynamic equations for plane magnetoacoustic waves [1] yield the following relations:

$$h_{\mu\nu x} = A_{\mu\nu} v_{\mu\nu z}, \qquad E_{\mu\nu y} = B_{\mu\nu} v_{\mu\nu z}, \qquad p_{\mu\nu} = Z_{\mu\nu} v_{\mu\nu z},$$
$$A_{\mu\nu} = \frac{H(u_{\mu\nu} - 1)\, u_{\mu\nu} k_{\mu\nu z}}{\omega \psi_\mu \beta_{\mu\nu}}, \qquad B_{\mu\nu} = \frac{-\omega A_{\mu\nu}}{c k_{\mu\nu z}},$$
$$Z_{\mu\nu} = \frac{(\operatorname{sign} k_{\mu\nu z})\, \rho_\mu a_\mu \sqrt{u_{\mu\nu}} \sin \alpha_{\mu\nu}}{\beta_{\mu\nu}},$$
$$\beta_{\mu\nu} = u_{\mu\nu} \cos \varphi - \sin \theta_{\mu\nu} \cos \alpha_{\mu\nu},$$
$$\alpha_{\mu\nu} = 90^\circ - (\theta_{\mu\nu} - \varphi), \qquad \alpha_{\mu\nu}' = 90^\circ - (\theta_{\mu\nu}' + \varphi),$$
$$\psi_\mu = \frac{H^2}{4\pi \rho_\mu a_\mu^2}, \qquad u_{\mu\nu} = \left(\frac{\omega}{k_{\mu\nu} a_\mu}\right)^2, \tag{1.1}$$

where v_z, p, and ρ are the velocity component in the z direction, the hydrodynamic pressure, and the density of the liquid, respectively, h is a small change in the magnetic field of the wave, Ey is the induced electric field, **k** is the wave vector, ω is the frequency, a is the ordinary velocity of sound in the fluid, c is the velocity of light, α is the angle between **k** and **H**, and u and ψ are the squares of the phase velocity and the density of the magnetic field in dimensionless form, respectively.

The angles $\theta_{\mu\nu}$ are related by Snell's law

$$k_{\mu\nu} \sin \theta_{\mu\nu} = k_{31} \sin \theta_{31}. \tag{1.2}$$

The phase velocities of the magnetoacoustic waves are determined by the dispersion relation

$$u_\mu^2 - (1 + \psi_\mu)\, u_\mu + \psi_\mu \cos^2 \alpha_\mu + i\omega\eta_\mu (u_\mu - 1) = 0,$$
$$\eta_\mu = c^2 / 4\pi\sigma_\mu a_\mu^2, \tag{1.3}$$

where σ_μ is the electrical conductivity of the medium.

In the case of well-conducting media $\omega\eta_\mu \ll 1$ and a weak magnetic field $\psi_\mu \ll 1$, we have from (1.3),

$$u_{\mu 1} = 1 + \psi_\mu \sin^2 \alpha_{\mu 1}, \qquad u_{\mu 2} = \psi_\mu \cos^2 \alpha_{\mu 2} - i\omega\eta_\mu. \tag{1.4}$$

If we retain only the principal terms, we have from (1.2),

$$\theta_{\mu\nu}' = \theta_{\mu\nu}, \qquad \frac{\sin \theta_{31}}{a_3} = \frac{\sin \theta_{\mu 1}}{a_\mu} = \frac{\sin \theta_{\mu 2}}{a_\mu \sqrt{\psi_\mu \sin^2 \varphi - i\omega\eta_\mu}}. \tag{1.5}$$

For a strong magnetic field $\psi_\mu \gg 1$ we have

$$u_{\mu 1} = \psi_\mu + \sin^2 \alpha_{\mu 1},$$
$$u_{\mu 2} = \cos^2 \alpha_{\mu 2} \left(1 - \frac{1}{\psi_\mu} \sin^2 \alpha_{\mu 2}\right),$$
$$\theta_{\mu\nu}' = \theta_{\mu\nu}, \qquad \sin \theta_{\mu 1} = \sqrt{\frac{\rho_3}{\rho_\mu}} \sin \theta_{31},$$
$$\sin \theta_{\mu 2} = \frac{1}{\sqrt{\psi_\mu}} \sin \varphi \sin \theta_{\mu 1}. \tag{1.6}$$

If we set the amplitude v_{31z} equal to unity, we can write the velocity field in the form

$$v_{3z} = -\exp[-i\gamma_{31}(z - d)] + \sum_{\nu=1}^{2} W_{3\nu}' \exp[i\gamma_{3\nu}(z - d)],$$
$$v_{1z} = -\sum_{\nu=1}^{2} W_{1\nu} \exp(-i\gamma_{1\nu} z),$$
$$v_{2z} = \sum_{\nu=1}^{2} [-W_{2\nu} \exp(-i\gamma_{2\nu} z) + W_{2\nu}' \exp(i\gamma_{2\nu} z)],$$
$$(\gamma_{\mu\nu} = k_{\mu\nu} \cos \theta_{\mu\nu}). \tag{1.7}$$

In these expressions we have omitted, for the sake of brevity, the common factor $[i(k_x x - \omega t)]$; $W_{3\nu}'$ and $W_{1\nu}$ are the amplitude reflection and transmission coefficients for layer, which are to be determined.

In accordance with Eqs. (1.7), the expressions for $h_{\mu x}$, $E_{\mu y}$, and p_μ are obtained by replacing the coefficients $W_{\mu\nu}$ in Eqs. (1.7) by $A_{\mu\nu} W_{\mu\nu}$, $B_{\mu\nu} W_{\mu\nu}$, $Z_{\mu\nu} W_{\mu\nu}$, respectively.

On the boundaries of the layer we have the conditions

$$v_{1z} = v_{2z}, \; h_{1x} = h_{2x}, \; E_{1y} = E_{2y}, \; p_1 = p_2 \quad \text{for} \quad z = 0,$$
$$v_{2z} = v_{3z}, \; h_{2x} = h_{3x}, \; E_{2y} = E_{3y}, \; p_2 = p_3 \quad \text{for} \quad z = d. \tag{1.8}$$

We thus obtain a set of eight equations for the coefficients $W_{\mu\nu}$.

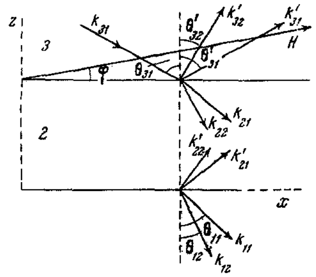
§2. If we restrict our attention to the principal terms only, the solution of this system for weak magnetic fields will be of the form

$$W_{31}' = V - a_2 \Delta^{-1} \sec \theta_{21} [(\rho_2 - \rho_3)(Z_1 \cos \gamma_{21} d -$$
$$- iZ_2 \sin \gamma_{21} d) W_{32}' - (\rho_1 - \rho_2) Z_1 W_{12}],$$
$$W_{11} = W - a_2 \Delta^{-1} \sec \theta_{21} [(\rho_2 - \rho_3) Z_3 W_{32}' -$$
$$- (\rho_1 - \rho_3)(Z_3 \cos \gamma_{21} d - iZ_2 \sin \gamma_{21} d) W_{12}],$$
$$W_{32}' = \delta^{-1} \rho_3^{-1} \Phi_2 [(\rho_2 - \rho_3)(\cos \gamma_{22} d -$$
$$- in \sin \gamma_{22} d)(1 + V) \operatorname{tg} \theta_{31} + (\rho_1 - \rho_2) W \operatorname{tg} \theta_{11}],$$
$$W_{12} = -\delta^{-1} \rho_1^{-1} \Phi_2 [(\rho_2 - \rho_3)(1 + V) \operatorname{tg} \theta_{31} +$$
$$+ (\rho_1 - \rho_2)(\cos \gamma_{22} d - im \sin \gamma_{22} d) W \operatorname{tg} \theta_{11}],$$
$$V = \Delta^{-1} [Z_2 (Z_1 - Z_3) \cos \gamma_{21} d - i (Z_2^2 - Z_1 Z_3) \sin \gamma_{21} d],$$
$$W = 2\Delta^{-1} Z_2 Z_3,$$
$$\Delta = Z_2 (Z_1 + Z_3) \cos \gamma_{21} d - i (Z_2^2 + Z_1 Z_3) \sin \gamma_{21} d,$$
$$\delta = (m + n) \cos \gamma_{22} d - i (1 + mn) \sin \gamma_{22} d,$$
$$m = \frac{a_3}{a_2} \left(\frac{\psi_3 \sin^2 \varphi - i\omega\eta_3}{\psi_2 \sin^2 \varphi - i\omega\eta_2}\right)^{1/2},$$

$$n=\frac{a_1}{a_2}\left(\frac{\psi_1\sin^2\varphi-i\omega\eta_1}{\psi_2\sin^2\varphi-i\omega\eta_2}\right)^{1/2},$$

$$\Phi_2=\frac{\psi_2\sin^2\varphi\sin\theta_{21}}{\sqrt{\psi_2\sin^2\varphi-i\omega\eta_2}},\qquad Z_\mu=\frac{\rho_\mu a_\mu}{\cos\theta_{\mu 1}},\qquad (2.1)$$

where V and W are the reflection and transmission coefficients of the layer in the absence of the magnetic field [2].



Thus, for oblique incidence of the magnetoacoustic wave on the layer, the reflection and transmission coefficients differ from the usual acoustic coefficients by terms of the order of $\psi_2(\psi_2\sin^2\varphi-i\omega\eta_2)^{-1/2}$.

In the case of perfectly conducting media the coefficient W_{12} is zero if

$$\operatorname{tg}\gamma_{21}d=mZ_1\left(\frac{1-M^2}{M^2Z_2^2-m^2Z_1^2}\right)^{1/2},$$

$$\operatorname{tg}\gamma_{22}d=\frac{Z_2}{mZ_1}\operatorname{tg}\gamma_{21}d\ ,$$

$$M=\frac{m^2-1}{n^2-1},\qquad m=\left(\frac{\rho_2}{\rho_3}\right)^{1/2},\qquad n=\left(\frac{\rho_2}{\rho_1}\right)^{1/2},$$

$$\gamma_{21}=\frac{\omega}{a_2}\cos\theta_{21},\qquad \gamma_{22}=\frac{\omega}{a_2\sqrt{\psi_2}\sin\varphi}\ .\qquad (2.2)$$

The equations given by (2.2) determine the thickness of the layer and the strength of the magnetic field for which there are no slow magnetoacoustic waves in medium 1. In particular, when $\rho_1<\rho_3<\rho_2$, real values of d are obtained for such a layer if

$$mZ_1/Z_2<M<1$$

The coefficient W_{32}' is zero if

$$\operatorname{tg}^2\gamma_{21}d=\Big(1+\zeta(1-2n^2M^2)+$$

$$+\sqrt{(\zeta-1)^2+4\zeta(n^2-1)(\zeta n^2-1)M^2}\Big)\times\Big(2\zeta(\zeta n^4M^2-1)\Big)^{-1},$$

$$\operatorname{tg}\gamma_{22}d=-\frac{Z_2}{nZ_1}\operatorname{tg}\gamma_{21}d,\qquad \zeta=\left(\frac{Z_2}{nZ_1}\right)^2.\qquad (2.3)$$

Therefore, when $\rho_1<\rho_3<\rho_2$, the thickness of the layer which will not reflect slow waves into medium 3 is determined by

$$\frac{Z_1}{nZ_2}<M<\frac{Z_1}{Z_2}\left(\frac{1+\zeta}{2}\right)^{1/2}\qquad(\zeta>1).$$

When media 3 and 1 are identical and $\gamma_{21}d=\pi$, $\gamma_{22}d=(2N+1)\pi$, so that

$$\sqrt{\psi_2}\sin\varphi\cos\theta_{21}=\frac{1}{2N+1},\qquad (2.4)$$

where N is a sufficiently large number, we have a perfectly transparent layer

$$W_{31}'=W_{32}'=W_{12}=0,\ W_{11}=-1\ .$$

If we set d = 0 in Eq. (2.1) we obtain the reflection and transmission coefficients of the separation boundary between the two media 3 and 1

$$W_{31}'=V_0-\frac{\psi_3}{2\Omega}a_3Y^2\cos\theta_{31},$$

$$W_{32}'=\frac{\psi_3}{\Omega}a_3Y\sin\varphi\sin\theta_{31},$$

$$W_{11}=W_0-\frac{\psi_1}{2\Omega}a_1Y^2\cos\theta_{11},$$

$$W_{12}=-\frac{\psi_1}{\Omega}a_1Y\sin\varphi\sin\theta_{11},\qquad V_0=\frac{Z_1-Z_3}{Z_1+Z_3},$$

$$W_0=\frac{2Z_3}{Z_1+Z_3},\qquad Y=\frac{2Z_3(\rho_1-\rho_3)}{(Z_1+Z_3)\rho_3}\sin\varphi\operatorname{tg}\theta_{11},$$

$$\Omega=a_1\sqrt{\psi_1\sin^2\varphi-i\omega\eta_1}+a_3\sqrt{\psi_3\sin^2\varphi-i\omega\eta_3}\ .\qquad (2.5)$$

At normal incidence, the magnetic field has no effect on the amplitude coefficients in the above approximation. If we include terms of the order of ψ, we have for this case

$$W_{31}'=V+\frac{\psi_2\cos^2\varphi}{2\Delta}\Big[Z_2\Big(\frac{a_2}{a_1}Z_2-Z_1\Big)W-$$

$$-\Big(\frac{a_2}{a_3}Z_2-Z_3\Big)(Z_2\cos k_{21}d-iZ_1\sin k_{21}d)(1+V)\Big],$$

$$W_{11}=W+\frac{\psi_2\cos^2\varphi}{2\Delta}\Big[Z_2\Big(\frac{a_2}{a_3}Z_2-Z_3\Big)(1+V)-$$

$$-\Big(\frac{a_2}{a_1}Z_2-Z_1\Big)(Z_2\cos k_{21}d-iZ_3\sin k_{21}d)W\Big],$$

$$W_{32}'=W_{12}=0\ .\qquad (2.6)$$

Consequently, at normal incidence and in a weak magnetic field, the reflected and refracted waves are of the same type as the incident wave.

For the separation boundary between two media we have from Eq. (2.6)

$$W_{31}'=V_0+\psi_3X,\quad W_{11}=W_0-\psi_3X,$$

$$X=Z_3\Big(\frac{a_3}{a_1}Z_3-Z_1\Big)(Z_1+Z_3)^{-2}\cos^2\varphi\ .\qquad (2.7)$$

§3. The solution of Eq. (1.8) for a strong magnetic field, which is accurate to within terms of the order of $(\psi)^{-1/2}$, is

$$W_{31}'=\frac{1}{\Delta}\left[n_2(n_1-n_3)\cos\gamma_{21}d-i(n_2^2-n_1n_3)\sin\gamma_{21}d\right],$$

$$W_{11}=\frac{2}{\Delta}n_2n_3\ ,$$

$$W_{12}=\frac{2n_2n_3\operatorname{tg}\varphi}{a_1\sqrt{\psi_1}\Delta\delta}\Big\{q(m_2\cos\gamma_{22}d-im_3\sin\gamma_{22}d)-$$

$$-rm_2\cos\gamma_{21}d]\cos(\theta_{11}-\varphi)+$$

$$+i\frac{rm_2}{\cos\theta_{21}}\Big[\Big(\frac{\rho_2}{\rho_1}\Big)^{1/2}\cos\varphi+\sin\theta_{21}\cos(\theta_{11}+\varphi)\Big]\sin\gamma_{21}d\Big\},$$

$$W_{32}'=-\frac{1}{m_2}\Big[W_{12}(m_2\cos\gamma_{22}d-im_1\sin\gamma_{22}d)+$$

$$+iW_{11}\frac{q}{a_1\sqrt{\psi_1}}\operatorname{tg}\varphi\cos(\theta_{11}-\varphi)\sin\gamma_{22}d\Big],$$

$$\Delta=n_2(n_1+n_3)\cos\gamma_{21}d-i(n_2^2+n_1n_3)\sin\gamma_{21}d,$$

$$q=m_2a_2=m_1a_1\ ,$$

$$\delta=m_2(m_1+m_3)\cos\gamma_{22}d-i(m_2^2+m_1m_3)\sin\gamma_{22}d_2,$$

$$r=m_2a_2-m_3a_3\ ,$$

$$n_\mu=\sqrt{\rho_\mu}\cos\theta_{\mu 1},\qquad m_\mu=\rho_\mu a_\mu,$$

$$\gamma_{21}=\frac{\omega\cos\theta_{21}}{a_2\sqrt{\psi_2}},\qquad \gamma_{22}=\frac{\omega}{a_2\sin\varphi}\ .\qquad (3.1)$$

If the thickness of the layer is much less than the wavelength of the fast magnetoacoustic wave, $\gamma_2 d\ll 1$, the formulas given by Eq. (3.1) are much simpler

$$W_{31}'=\frac{n_1-n_3}{n_1+n_3}-2i\frac{n_2^2-n_1^2}{(n_1+n_3)^2}\gamma_{31}d,$$

$$W_{11}=\frac{2n_3}{n_1+n_3}+2i\frac{n_2^2+n_1n_3}{(n_1+n_3)^2}\gamma_{31}d\ ,$$

$$W_{32}'=\frac{\beta}{\delta\sqrt{\psi_1}}\left[r(m_2\cos\gamma_{22}d-im_1\sin\gamma_{22}d)-qm_2\right],$$

$$\beta = \frac{2n_3 \operatorname{tg} \varphi \cos(\theta_{11} - \varphi)}{a_1 (n_1 + n_3)},$$

$$W_{12} = \frac{\beta}{\delta \sqrt{\psi_1}} [q (m_2 \cos \gamma_{22} d - i m_3 \sin \gamma_{22} d) - r m_2],$$

$$\gamma_{31} = \frac{\omega \cos \theta_{31}}{a_3 \sqrt{\psi_3}}. \qquad (3.2)$$

The slow waves will now appear even in the case of normal incidence. According to Eq. (3.2), they are absent only when

$$\gamma_{22} d = (2N + 1) \pi, \; 2\rho_2 a_2^2 = \rho_1 a_1^2 + \rho_3 a_3^2, \qquad (3.3)$$

i.e., when the thickness of the layer is equal to an odd number of half-waves of the second type, and the modulus of elasticity of medium 2 is equal to the arithmetic mean of the moduli of media 3 and 1.

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